Lecture 28

Euler Tours, More Notations

Euler Tours

Theorem: A connected graph G has an Euler tour if and only if all the vertices are of even degree.

Proof: $(\implies)$ Let W be an Euler tour.

Clearly, W visits all the vertices of G a certain number of times.

Suppose v is not the starting and ending vertex of W and W visited it k times.

It means W entered v exactly k times and exited v exactly k times through $2k$ different edge as W contains distinct edges only.

Additionally, v cannot have any other edge incident on it apart from these $2k$ edges as W contains all the edges. Hence, $degree(v) = 2k$.

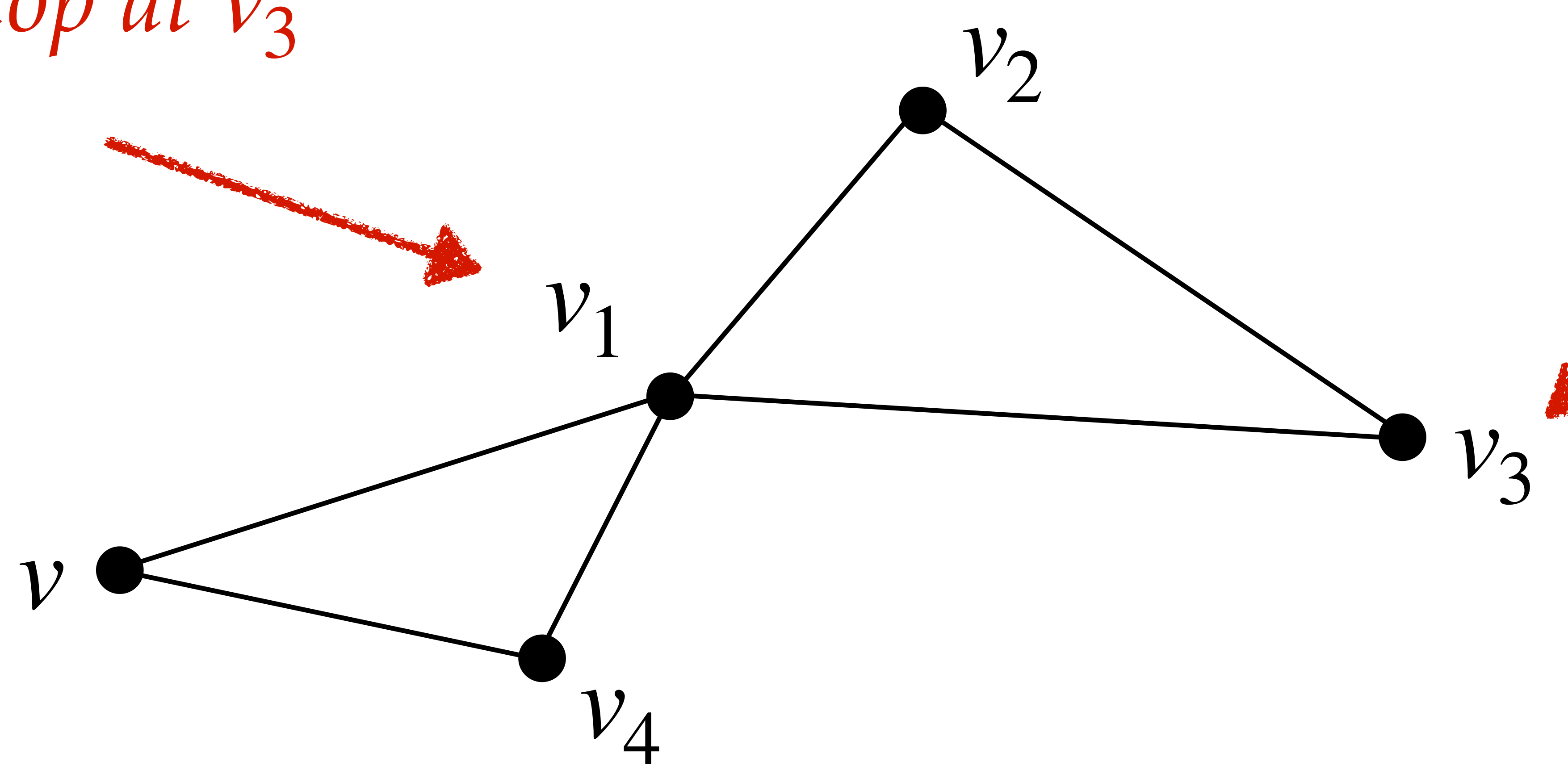
The degree of the starting and ending vertex that gets visited k' in the middle of the walk can be similarly shown to be $1 + 2k' + 1$, which is even. …

Euler Tours

$(\Longleftarrow)$ Take any vertex v and move to a vertex v_1 through the edge $\{v, v_1\}$, then move to a vertex v_2 through a new edge $\{v_1, v_2\}$, and so on...

Won't stop at v_1 for the same reason it didn't stop at v_3

Won't stop at v_3 as v_3 has even degree and so far we've seen odd number of edges incident upon v_3



In this process, we will not run out of edges before returning to v because if we can enter a vertex we can also exit as it has an even degree.

...

Euler Tours

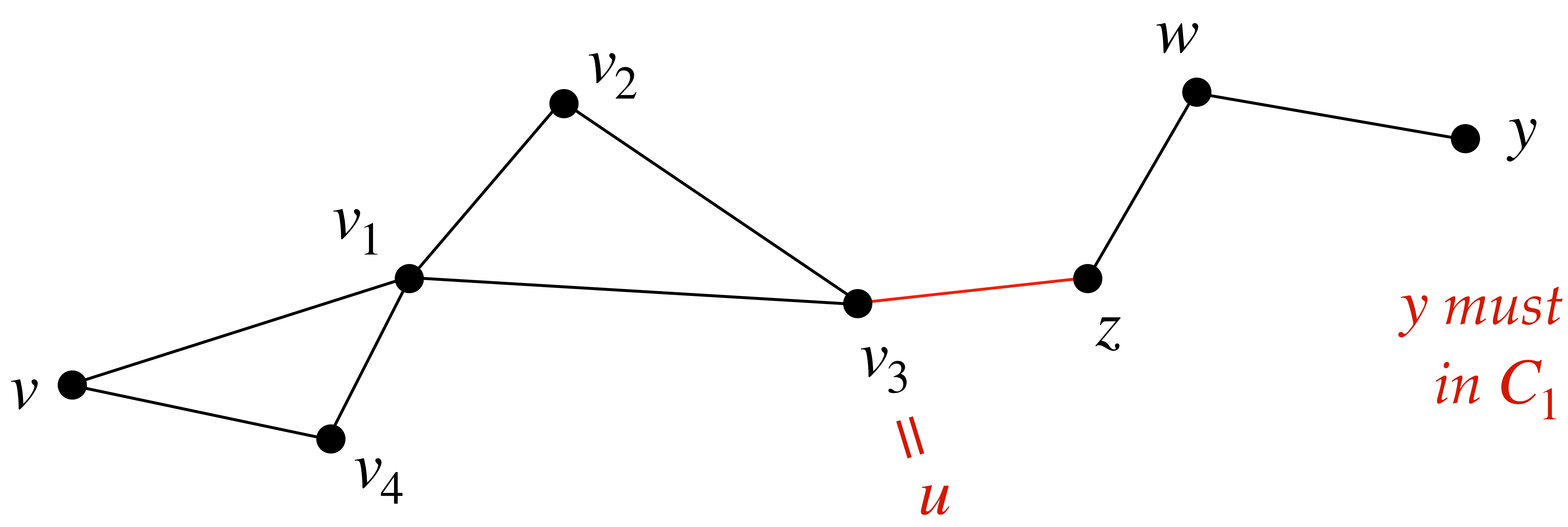
Let C_1 be the closed walk formed in this process. If C_1 is an Euler tour, we are done.

If C_1 is not an Euler tour, then pick a vertex u in C_1 whose some incident edges are not in C_1.

Why should such a u exist?

Suppose all vertices in C_1 have all their incident edges in C_1 as well.

Then, there must exist some vertex y not in C_1, otherwise C_1 will be an Euler tour.



y must have a path to some vertex in C_1 as the graph is connected

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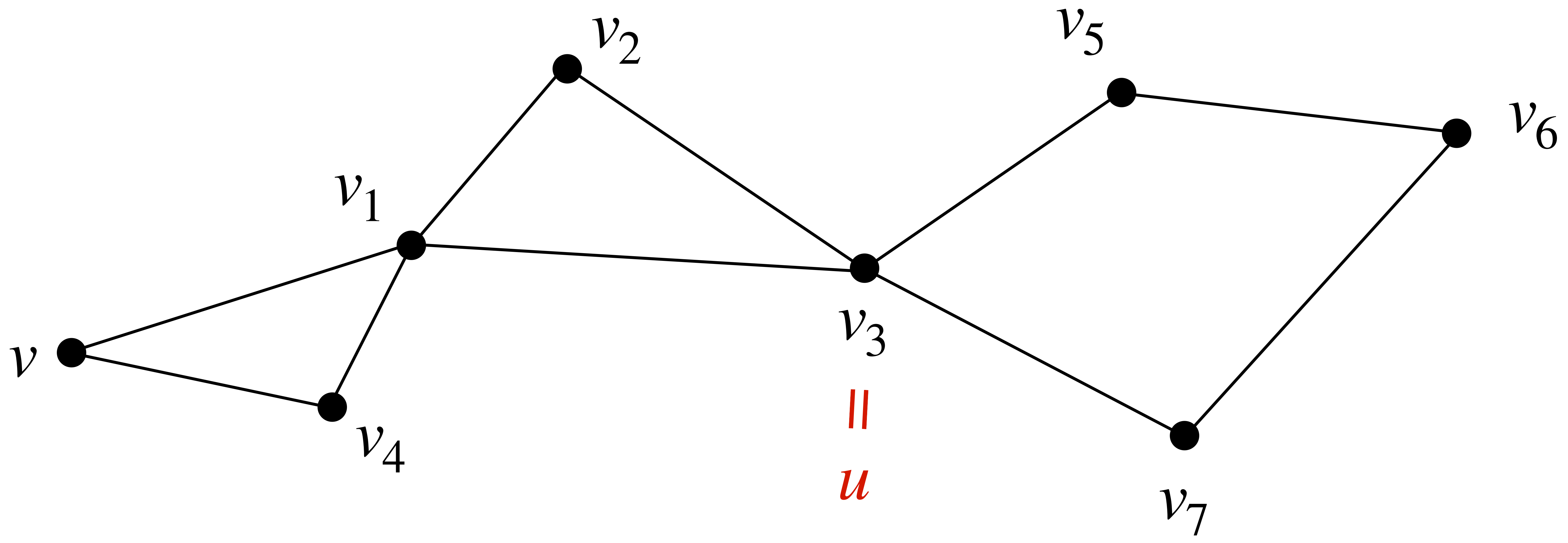
Euler Tours

Delete all the edges of C_1 from G.

The resulting graph will again have even degrees.

Start the same process from u in the new graph. Let C_2 be the closed walk that is formed from this process.



Let C be the closed walk formed by concatenating C_1 with C_2.

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Euler Tours

If C is an Euler tour, then we are done.

If C is not an Euler tour, then pick a vertex w in C whose some incident edges are not in C.

Repeat the same procedure. Get another closed walk C_3 with a common vertex w with C.

Form a larger closed walk with C and C_3 and so on ...

Since G is finite, this process will stop after a finite number of steps, and the resulting closed walk has to be an Euler tour.

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More Notations

- For graphs $G = (V, E)$ and $G' = (V', E')$, if $V' \subseteq V$ and $E' \subseteq E$, then G' is a **subgraph** of G.
- For a graph $G = (V, E)$ and $U \subseteq V$, $G - U$ is obtained from G by deleting all the vertices in U and their incident edges.
- For a graph $G = (V, E)$ and a subset F of $[V]^2$, we write $G - F = (V, E \backslash F)$ and $G + F = (V, E \cup F)$.
- Complement $\overline{G}$ of a graph $G = (V, E)$ is the graph on V with edge set $[V]^2 \backslash E$.